The Ideal Theory of C^*-Algebras

by

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Abstract

For a C^*-algebra A, the topological relation between the pure states space $\mathrm{P}(A)$, the dual space $\hat{A}$ and the primitive ideals space $\mathrm{Prim}(A)$ has been investigated by a lot of mathematicians, for example, J.M.G. Fell and Jacques Dixmier.

In this thesis, we base on Archbold's work, giving a survey on the topological properties of primal ideals and generalizing the classical results.

摘要

對一 C^* 代數 A, 純態空間 PS(A), 偶空間 $\hat{A}$ 及原理想空間 Prim(A) 之間的拓撲性質, 已經被很多作者如 J.M.G. Fell, Jacques Dixmier 等研究過了.

本論文將會展示 Archbold 的工作,探討 Primal 理想空間的拓樸性質及推廣經典的結果.

Introduction

It is well known that each abelian C^*-algebra A is $*$-isomorphic to $\mathcal{C}_0(\Omega)$, the C^* algebra of all complex-valued continuous functions defined on Ω vanishing at infinity, where Ω denotes the characters space of A. In abelian case, the structures of C^* algebras are simple and the three objects, namely, pure states, irreducible representations, primitive ideals are identical.

The structures of non-abelian C^*-algebras are much more complicated and little is known today. The three objects cannot be identified with each other any more, however, they are still closely related. In fact, we have the following result: Let A be a C^*-algebra. We denote the pure states space of A , the space of unitary equivalent classes of irreducible representations of A and the primitive ideal space of A by $\mathrm{PS}(A)$, $\hat{A}$ and $\mathrm{Prim}(A)$ respectively. We also endow $\mathrm{PS}(A)$, $\hat{A}$ and $\mathrm{Prim}(A)$ with the weak*, Fell and hull-kernel topologies respectively. Given $\phi \in \mathrm{PS}(A)$, we denote π_ϕ the irreducible representation associated with ϕ via GNS construction. Let $\theta_1 : \mathrm{PS}(A) \longrightarrow \hat{A}$ be the map defined by $\theta_1(\phi) = [\pi_\phi]$ and $\theta_2 : \hat{A} \longrightarrow \mathrm{Prim}(A)$ be the map defined by $\theta_2([\pi]) = \ker[\pi]$. Then both θ_1 and θ_2 are surjective, open and continuous. [2]

Primitive ideals space is a little bit restrictive. In fact, we have a nice extension of our discussion on the structure theory of C^* algebras. In analogue, the objects pure states, irreducible representations and primitive ideals are replaced by factorial states, factor representations and primal ideals. In the paper [6], Archbold first investigated the algebraic relationship over these new objects, then, in his next paper [7], he also gave suitable topologies on the primal ideal space such that the canonical maps between these objects are open and continuous (an analogue of the classical result stated in above).

Lastly, the minimal primal ideals space will be used as a base space in decomposing a C^* algebra into the full algebra of continuous field of operators in contrast to Fell's result in which the primitive ideals space is used as the base space. [3] 3.4.11.

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Contents

Chapter 1

Preliminaries

This chapter is an introduction. We will first define C^*- algebras and go through their basic properties concerning pure states, irreducible representations and primitive ideals.

1.1 Definitions

Definition 1.1.1. A Banach $*$-algebra A is a Banach space together with two algebraic operations, namely, the multiplication $(a, b) \mapsto ab$ from $A \times A$ to A and involution $a \mapsto a^*$ from A to A such that:
For any $a, b, c \in A$ and $\lambda \in \mathbb{C}$,
(i) $(ab)c = a(bc)$, and
(ii) $(a + b)c = ac + bc$ and $c(a + b) = ca + cb$, and
(iii) $\lambda(ab) = (\lambda a)b = a(\lambda b)$, and
(iv) $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$, and
(v) $(ab)^* = b^* a^*$, and
(vi) $(a^*)^* = a$, and
(vii) $\| a \| = \| a^* \|$, and
(viii) $\| ab \| \leq \| a \| \| b \|$.
If, in addition, that $\| a^* a \| = \| a \|^2$, we call A a C^*-algebra.

Definition 1.1.2. Let A be a Banach $*$-algebra. If there exists an element $1 \in A$ such that $a1 = 1a = a$ for all $a \in A$ and $\| \mathbf{1} \| = 1$, we say that A is unital with unit 1.

Remark:
If A is unital, the unit is unique.

Definition 1.1.3. Let A be a Banach $*$-algebra (C^*-algebra repsectively) and B a non-empty subset of A. If B itself is a Banach $*$-algebra (C^*-algebra) with the restricted algebraic operations and norm from A, then B is said to be a Banach $*$-subalgebra (C^*-subalgebra) of A.

Definition 1.1.4. Let A be a non-unital Banach $*$-algebra. We can adjoint a unit to it. Let $\widetilde{A} = A \oplus \mathbb{C}$ as a vector space. Then define multiplication and involution on it by

$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$ and

$(a, \lambda)^* = (a^*, \overline{\lambda})$.

We also define the norm by $\| (a, \lambda) \| = \| a \| + |\lambda|$.

Then $\widetilde{A}$ becomes a unital Banach $*$-algebra and A is embedded into $\widetilde{A}$ by the map $a \mapsto (a, 0)$. $\widetilde{A}$ is said to be the unitalization of A.

Remark:

If A is a C^*-algebra, then $\widetilde{A}$ with the norm defined above is NOT a C^* norm, since it does not satisfy the C^* equality $\| a^*a \| = \| a \|^2$.

However, there is always a norm on $\widetilde{A}$ such that $\widetilde{A}$ becomes a unital C^*-algebra, and, when this norm is restricted on A, it coincides with the original norm on A. (Such norm is necessarily unique by the property of C^*- algebra.) [1] [Theorem 2.1.6]. We call $\widetilde{A}$ with this norm the C^* unitalization of A.

Definition 1.1.5. Let A be a Banach $*$-algebra. Let $a \in A$. Then

(i) a is said to be self-adjoint if $a^* = a$,

(ii) a is said to be normal if $a^*a = aa^*$,

(iii) a is said to be isometry if $a^*a = 1$, and co-isometry if $aa^* = 1$, provided that A is unital.

(iv) a is said to be unitary if $a^*a = aa^* = 1$, provided that A is unital.

(v) a is said to be a projection if $a = a^* = a^2$.

We denote the set of all self-adjoint elements of A by A_{sa}.

Remark:

For each element $a \in A$, there exists uniquely self-adjoint elements $b, c \in A$ such that $a = b + ic$. We call b, c the real and imaginary parts of a and denote them by $b = \Re(a)$, $c = \Im(a)$. In fact, they are given by $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$.

Definition 1.1.6. Let A be a unital Banach $*$-algebra and $a \in A$. a is said to be invertiable if there exists $b \in A$ such that $ab = ba = 1$. In this case, b is uniquely determined by a. Theorefore we denoted such b by a^{-1}, called the inverse of a.

Definition 1.1.7. Let A be a unital Banach $*$-algebra and $a \in A$. We define the spectrum of a by $\sigma_A(a) = \{\lambda \in \mathbb{C} |\ \lambda 1 - a \text{ is not invertivble}\}$. If there is no confusion, we simply denote it by $\sigma(a)$.
If A is non-unital, we define the spectrum of a to be $\sigma_A(a) = \sigma_{\tilde{A}}((a, 0))$.

Remark:
The $\sigma(a)$ is a non-empty and compact subset of $\mathbb{C}$

[1][Lemma 1.2.4][Theorem 1.2.5]

Definition 1.1.8. Let A be a Banach $*$-algebra. Let $I \subseteq A$. Suppose that I is a vector subspace of A.
(i) If in addition, $ab \in I$ for all $a \in A$ and $b \in I$, I is said to be a left ideal of A.
(ii) If in addition, $ba \in I$ for all $a \in A$ and $b \in I$, I is said to be a right ideal of A. If both (i), (ii) are satisfied, I is said to be a two-sided ideal of A. An ideal is said to be closed if it is a closed subset of A in the norm topology. We denote $a + I = \{a + b |\ b \in I\}$. From now on, the word "ideal" refer to "closed two-sided ideal" unless otherwise specified.
Remark
If A is a C^*-algebra, it is automatical that $a^* \in I$ for any $a \in I$. [1] [Theorem 3.1.3]

Definition 1.1.9. Let A be a C^*-algebra and I an ideal of A. We define $A/I = \{a + I|\, a \in A\}$ with the following algebraic operations and norm:
(i) $(a+I)+(b+I) = (a+b)+I$,
(ii) $\lambda(a+I) = (\lambda a)+I$,
(iii) $(a+I)(b+I) = ab+I$,
(iv) $(a+I)^* = (a^*)+I$,
(v) $\| a+I \| = \inf_{b\in I} \| a+b \|$.
Then A/I becomes a C^*-algebra.

Definition 1.1.10. Let A be a C^*-algebra. Let a be a self-adjoint element of A. a is said to be positive if $\sigma(a) \subset [0, +\infty)$.

Remark:
It is well-known that a is positive if and only if there exists $b \in A$ such that $a = b^*b$. [1] [Theorem 2.2.4]

Definition 1.1.11. Let A be a C^*-algebra. A linear functional ϕ on A is said to be positive if $\phi(a) \geq 0$ for any positive element a.
A positive linear functional ϕ is said to be a state if $\| \phi \| = 1$.
A state ϕ is said to be pure if for any positive linear functional ψ satisfying $\psi \prec \phi$ (i.e. $\phi(a) - \psi(a) \geq 0$ for any positive element a), there exists $t \in [0,1]$ such that $\psi = t\phi$.)
We denote the set of all states and pure states of A by S(A) and PS(A) respectively and endow S(A) with the weak* topology. Note that S(A) is locally compact and Hausdorff. If A is unital, S(A) is compact Hausdorff.

Remark:

We denote the space of quasi-states (i.e., positive linear functionals with norm ≤ 1) by $\mathrm{QS}(A)$. The extreme points of $\mathrm{QS}(A)$ are the zero functional and the pure states. [1] [Theorem 5.1.8]. By Krein-Milman theorem, it follows that $\mathrm{QS}(A)$ is the weak* closed convex hull of $\{0\} \cup \mathrm{PS}(A)$. [1] [Corollary 5.1.9]

Definition 1.1.12. Let A and B be C^*-algebras. A $*$-homomorphism θ from A into B is a linear map which also preserves multiplication and involution, i.e. $\theta(ab) = \theta(a)\theta(b)$ and $\theta(a^*) = (\theta(a))^*$.
A representation of a C^*-algebra A is a pair (π, H) such that $\pi : A \longrightarrow \mathrm{B}(H)$ is a $*$-homomorphism, where $\mathrm{B}(H)$ denotes the C^*-algebra of all bounded linear operators on the Hilbert space H. For simplicity, we will abuse the use of terminology and call π a representation of A.

Remark:

If ϕ is a $*$-homomorphism from C^*-algebra A into C^*-algebra B, it must be norm-decreasing. [2] [1.3.7] [1] [Theorem 2.1.7] If ϕ is injective, then it must be an isometry. [2] [1.8.1] [1] [Theorem 3.1.5].

Definition 1.1.13. Let (π, H) be a representation of a C^*-algebra A.
(i) π is said to be non-degenerate if $H = [\pi(A)H]$, where $[\pi(A)H]$ denotes the closure of the linear span of $\{\pi(a)\xi \,|\, a \in A,\, \xi \in H\}$.
(ii) π is said to be cyclic if there exists $\xi \in H$ such that $H = [\pi(A)\xi]$. Such vector ξ is called a cyclic vector.
(iii) A subspace K of H is said to be invariant under π if $\pi(a)\xi \in K$ for each $a \in A$ and each $\xi \in K$.

(iv) π is said to be algebraically irreducible if there is no non-trival invariant subspace.

(v) π is said to be topologically irreducible if there is no non-trival closed invariant subspace.

Remark:

(i) is equivalent to the condition:

For each non-zero $\xi \in H$, there exists $a \in A$ such that $\pi(a)\xi$ is non-zero.

It seems that condition (iv) is stronger than the condition (v), however, they are in fact equivalent. This is the well-known Kadison's Transitivity Theorem [1] [Theorem 5.2.3].

An irreducible representation is automatically cyclic and a cyclic representation is non-degenerate.

Definition 1.1.14. Let $(\pi_1, H_1), (\pi_1, H_2)$ be representations of A. They are said to be unitary equivalent if there exists a unitary map $u : H_1 \longrightarrow H_2$ such that $u\pi_1(a) = \pi_2(a)u$ for all $a \in A$.

From now on, we denote the set of all unitary equivalent classes of all irreducible representations of A by $\widehat{A}$. Here, we must justify that $\widehat{A}$ is a set (since the class of all irreducible representations of A is **NOT** a set). We follow J.M.G. Fell [4] and fix a Hilbert space H of sufficient large cardinality. By an irreducible representation π, we mean (π^K, K) is an irreducible representation, where $K = [\pi(A)H]$ and π^K is the $*$-homomorphism from A to $\mathrm{B}(K)$ such that $\pi^K(a) = \pi(a)|_K$.

Moreover, we will not distinguish an irreducible representation π and its unitary equivalent class $[\pi]$. We abuse the notation and simply denote $[\pi]$ by π unless otherwise specified.

Definition 1.1.15. Let A be a C^*-algebra. An ideal P of A is said to be primitive if there exists an irreducible representation π of A such that $P = \ker(\pi)$.

Remark:

A primitive ideal P is automatically a prime ideal. [1] [Theorem 5.4.5]

Definition 1.1.16. Let A be a C^*-algebra. We denote the set of all primitive ideals of A by Prim(A). We endow Prim(A) with the hull-kernel topology as follows:
If I is a closed two-sided ideal, we define hull(I)=$\{P \in \mathrm{Prim}(A) | I \subseteq P\}$. If S is a subset of Prim(A), we define $\ker(S) = \cap_{P \in S} P$. We also define $\overline{S} = \mathrm{hull}(\ker(S))$. It can be proved that the map $S \mapsto \overline{S}$ satisfies the Kuratowski's axiom of closure operator. [2] [3.1.1], [1] [Theorem 5.4.6] Hence it defines a topology on Prim(A), called the Jacobson topology.

Definition 1.1.17. Let A be a C^*-algebra and $\theta : \widehat{A} \longrightarrow \mathrm{Prim}(A)$ be the canonical quotient map $\theta(\pi) = \ker(\pi)$. We endow $\widehat{A}$ with the smallest topology such that θ is continuous. Such topology is called Fell's topology and the topological space $\widehat{A}$ is known as the structure space of A.

Remark:

The Fell's topology is locally compact in the sense that for each point $\pi \in \widehat{A}$ and each open neighborhood U of π, there exists a compact neighborhood V of π such that $V \subseteq U$. [2] [Proposition 3.3.8]
If A is unital, the Fell's topology is compact. [2] [Proposition 3.1.8]
Since Fell's topology is an initial topology and the canonical map is surjective, the canonical map is thus continuous, open and closed. Therefore the closure operator on $\widehat{A}$ is given by
$\overline{S} = \{\pi \in \widehat{A} | \ker(\pi) \supseteq \cap_{\pi' \in S} \ker(\pi')\}$

1.2 Topological Properties of the Structure Spaces of C^*-Algebras

Proposition 1.2.1. Let A be a C^*-algebra and I an ideal of A.

Denote

$\mathrm{Prim}_I(A) = \{P \in \mathrm{Prim}(A) |\ I \subseteq P\}$,

$\mathrm{Prim}^I(A) = \{P \in \mathrm{Prim}(A) |\ I \not\subseteq P\}$,

$\widehat{A_I} = \{\pi \in \widehat{A} |\ I \subseteq \ker(\pi)\}$,

$\widehat{A^I} = \{\pi \in \widehat{A} |\ I \not\subseteq \ker(\pi)\}$.

We have the following commutation diagrams, where the vertical maps denote the quotient maps and the horizontal maps denote the canonical maps described below. Moreover, the horizontal maps are homeomorphic.

$$\begin{array}{ccccccc} \widehat{A_I} & \longrightarrow & \widehat{A/I} & & \widehat{A^I} & \longrightarrow & \widehat{I} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Prim}_I(A) & \longrightarrow & \mathrm{Prim}(A/I) & & \mathrm{Prim}^I(A) & \longrightarrow & \mathrm{Prim}(I) \end{array}$$

The horizontal maps are defined as follow:

$\widehat{A_I} \longrightarrow \widehat{A/I}$, $\pi \mapsto \pi'$ where $\pi'(a+I) = \pi(a)$,

$\widehat{A^I} \longrightarrow \widehat{I}$, $\pi \mapsto \pi|_I$,

$\mathrm{Prim}_I(A) \longrightarrow \mathrm{Prim}(A/I)$, $P \mapsto \{x + I |\ x \in P\}$,

$\mathrm{Prim}^I(A) \longrightarrow \mathrm{Prim}(I)$, $P \mapsto P \cap I$.

Proof. Refer to [2] [3.2.1] $\square$

Proposition 1.2.2. Let A be a C^*-algebra. For each $a \in A$, the function $\pi \mapsto \| \pi(a) \|$ defined on $\widehat{A}$ is lower-semi continuous.

Proof. Let $\epsilon > 0$. We need to show $S = \{\pi \in \widehat{A} |\ \| \pi(a) \| \leq \epsilon\}$ is closed. Note that $\| \pi(a^*a) \| = \| \pi(a) \|^2$, so we may assume that a is positive. Let $\pi \in \overline{S}$. If $\| \pi(a) \| > \epsilon$, we choose α such that $\| \pi(a) \| > \alpha > \epsilon$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the continuous function which is zero on $(-\infty, \epsilon]$, one on $[\alpha, +\infty)$ and linear on (ϵ, α). Note that $\pi(f(a)) = f(\pi(a)) \neq 0$, hence $f(a) \notin \ker(\pi)$. On the other hand, if $\pi' \in S$, then $\pi'(f(a)) = f(\pi'(a)) = 0$, i.e. $f(a) \in \cap_{\pi' \in S} \ker(\pi')$. But $\pi \in \overline{S}$ implies $\cap_{\pi' \in S} \ker(\pi') \subseteq \ker(\pi)$. A contradiction. □

Proposition 1.2.3. Let A be a C^*-algebra. For each $a \in A$, there exists an irreducible representation π of A such that $\| \pi(a) \| = \| a \|$.

Proof. See [1] [Theorem 5.1.12] □

Remark:

Let $P = \ker(\pi)$, we have $\| a + P \| = \| \pi(a) \|$. This can be proved by using the fact that $x + P \mapsto \pi(x)$ is an injective $*$-homomorphism on A/P, hence an isometry.

Proposition 1.2.4. Let A be a C^*-algebra. For each $a \in A$ and each $\epsilon > 0$, the set $\{\pi \in \widehat{A} | \; \| \pi(a) \| \geq \epsilon\}$ is compact.

Proof. Let $S = \{\pi \in \widehat{A} | \; \| \pi(a) \| \geq \epsilon\}$. Let $\{F_\lambda\}$ be a descending chain of non-empty closed subsets of S. We need to show $\cap F_\lambda \neq \emptyset$ and this will prove the completeness of S. [5] [P.163 H]. For each λ, let $I_\lambda = \cap_{\pi \in F_\lambda} \ker(\pi)$. $\{I_\lambda\}$ is an increasing chain of ideals and $\| a + I_\lambda \| \geq \epsilon$ for each λ. Let $J = \overline{\cup_\lambda I_\lambda}$. For each $x \in \cup_\lambda I_\lambda$, there exists λ such that $x \in I_\lambda$, so $\| a + x \| \geq \| a + I_\lambda \| \geq \epsilon$. Hence $\| a + J \| \geq \epsilon$. Choose a primitive ideal P containing J such that $\| a + P \| = \| a + J \|$. Lastly, choose an irreducible representation π with $\ker(\pi) = P$, then $\pi \in \cap_\lambda F_\lambda$. □

Let A be a C^*-algebra and ϕ a state of A. Define a positive sesquilinear form on $A \times A$ by $(x, y) \mapsto \phi(y^*x)$. Let $N = \{x| \, \phi(x^*x) = 0\}$ which is a closed left ideal of A. Regarding A as a vector space, on the quotient space A/N, we define an inner product by $< x + N, y + N >= \phi(y^*x)$ and let H be the completion of the inner product space A/N. For each $a \in A$, we define $\pi(a)$ to be a bounded linear operator on A/N by $\pi(a)(b + N) = ab + N$. Then we extend $\pi(a)$ to be a bounded linear operator on H (still denoted by $\pi(a)$). It is routine to show that the map $a \mapsto \pi(a)$ is a *-homomorphism from A to $\mathrm{B}(H)$. Such (π, H) is known as the representation associated with ϕ.

The above construction is known as GNS construction.

Theorem 1.2.5. Using the above notation, The representation (π, H) of A is cyclic with unit cyclic vector ξ such that:

(i) $\phi(a) =< \pi(a)\xi, \xi >$, and

(ii) $\pi(a)\xi = a + N$

Proof. Now we construct the cyclic vector ξ. Note that the map $a+N \mapsto \phi(a)$ is a well defined, bounded linear functional. It can be extended to a bounded linear functional ρ on H. By Riesz representation theorem, there exists a vector $\xi \in H$ such that $\rho(\eta) =< \eta, \xi >$, $\eta \in H$. Hence $< a + N, \xi >= \rho(a + N) = \phi(a)$. Let $a, b \in A$. Consider $< b+N, \pi(a)\xi >=< a^*b+N, \xi >= \phi(a^*b) =< b+N, a+N >$. Hence $\pi(a)\xi = a + N$. As A/N is dense in H, the representation π is cyclic with cyclic vector ξ.

Let $\{u_\lambda\}$ be an approximate identity of A. Then $\pi(u_\lambda)$ is an approximation identities of $\pi(A)$ and hence it converges to I_H in $\mathrm{B}(H)$ strongly. (since π is cyclic) We have $\| \xi \|^2 = \lim_\lambda < \pi(u_\lambda)\xi, \xi >= \lim_\lambda \phi(u_\lambda) = 1$. □

1.3 Topological Relations Between $\mathrm{P}(A)$, $\hat{A}$ and $\mathrm{Prim}(A)$

Theorem 1.3.1. Let A be a C^*-algebra. Then the canonical map $\mathrm{P}(A) \to \widehat{A}$, $\phi \mapsto \pi_\phi$ is open and continuous. Hence the topology on $\hat{A}$ is the quotient topology induced by the canonical map.

Proof. Refer to [2] [3.4.11] $\square$

Remark:

The canonical map from $\hat{A}$ onto $\mathrm{Prim}(A)$ is open and continuous.

Chapter 2

Topological Properties of the Primal Ideal Spaces of C^*-Algebras

In this chapter, we will first give the definition of primal ideals of a C^*-algebra. Then it will be endowed with two important topologies, namely, the weak topology and the strong topology. A main result in this chapter is to generalize the classical relation between pure states, irreducible representation and primitive ideals.

It is known that if ϕ is a pure state, then π_ϕ and $\ker(\pi_\phi)$ are irreducible representation and primitive ideal respectively. Moreover, the canonical maps $\phi \mapsto \pi_\phi$, $\pi_\phi \mapsto \ker(\pi_\phi)$ are all open and continuous. When time goes on, mathematicians start to investigate a class of states which is contains than pure states, namely, the class of factorial states. It is natural to ask whether there is an analogue of the classical result above. Archbold proved that for a C^*-algebra A, a state ϕ is the weak* limit of factorial states if and only if $\ker(\pi_\phi)$ is a primal ideal of A. [6] [Theorem 3.5]. In order to start a deeper investigation, Archbold defined two topologies on the primal ideal space and generalized some classical results of pure states and primitive ideals (see [7]). We base on the work of Archbold and give a survey on the topological properties of primal ideals.

Definition Let A be a C^*-algebra and I an closed two sided ideal of A. I is said to be a primal ideal of A if:
For finitely many ideals J_i of A, $i = 1, 2, \ldots, n$, if $J_1 J_2 \cdots J_n (= \bigcap_{i=1}^{n} J_i) = 0$, then $J_i \subseteq I$ for some i.

Remark:
It is natural to ask whether n can be replaced by a fixed integer, for example, $n = 2$. The answer is no and is illustrated by a counter-example given by Archbold. [6] [P.59]
We also remark that each primitive ideal is a prime ideal [1] [Theorem 5.4.5] and clearly each prime ideal is a primal ideal.

From now on, we denote the set of all closed two-sided ideals and the set of all primal ideals of A by Id(A) and Primal(A) respectively.

2.1 Topologies on Id(A)

We now define two topologies on Id(A), namely, the weak topology (τ_w) and the strong topology (τ_s). The weak topology is a natural generalization of the hull-kernel topology on Prim(A) and the strong topology is in fact the one defined by J.M.G. Fell in his paper [3].

Definition 2.1.1. Let A be a C^*-algebra. Let $\mathcal{F}$ be a finite collection of ideals of A ($\mathcal{F}$ may be empty). We define $\mathcal{U}(\mathcal{F})$=$\{$I$\in Id(A)|$J$\not\subseteq$I, $\forall$J$\in \mathcal{F}\}$. Then all such $\mathcal{U}(\mathcal{F})$ constitutes a base of some topology on $Id(A)$. This topology is denoted by τ_w and is called the weak topology on $Id(A)$.

Proof. Choose $\mathcal{F}$ to be the empty collection and $\{0\}$, we immediately observed that $Id(A)$ and the empty set are contained in τ_w. Let $\mathcal{U}(\mathcal{F}_1)$ and $\mathcal{U}(\mathcal{F}_2)$ be two such sets.Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$,which is still a finite collection of ideals. Note that $\mathcal{U}(\mathcal{F}_1) \cap \mathcal{U}(\mathcal{F}_2) = \mathcal{U}(\mathcal{F})$. This shows that such $\mathcal{U}(\mathcal{F})$ constitutes a base. □

Remark:

It is obvious that when τ_w is restricted on Prim(A), it is precisely the Jacobson topology.

Definition 2.1.2. Let A be a C^*-algebra. For each $a \in A$, we define a function on Id(A) by $N_a(I) = \| a + I \|$, ($I \in$Id(A)). We define the strong (τ_s) topology on Id(A) to be the weakest topology such that all the functions N_a are continuous.

Remark:

Let $I \in$Id(A). A local base of I is given by $\{J \in \text{Id}(A) \mid | \| a_i + J \| - \| a_i + I \| | < \epsilon \text{ for all } i = 1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, $a_i \in A$, $\epsilon > 0$.

A net I_λ in Id(A) τ_s converges to I if and only if $\| a + I_\lambda \| \to \| a + I \|$ for each $a \in A$.

Proposition 2.1.3. The space Id(A) is compact and Hausdorff in τ_s topology.

Proof. The space is clearly Hausdorff. For, if not, then there exists a net $\{I_\lambda\}$ which τ_s converges to two distinct points J_1 and J_2. If there exists $a \in J_1 \backslash J_2$, then $0 = \| a + J_1 \| = \lim_\lambda \| a + I_\lambda \| = \| a + J_2 \| > 0$ contradiction ! Therefore $J_1 \subseteq J_2$ and similarly we can show $J_2 \subseteq J_1$. It is a contradiction.

Let I_λ be a net in Id(A). We claim that there exists a subnet $I_{\lambda'}$ such that $\lim_{\lambda'} \| I_{\lambda'}(a) \|$ is convergent for each $a \in A$. For each λ, we define a function on the closed unit ball S of A by $f_\lambda(a) = \| a + I_\lambda \|$. Note that $f_\lambda \in [0,1]^S$ and $[0,1]^S$

is compact by Tychonoff theorem. Hence there exists a subnet $f_{\lambda'}$ and $f \in [0,1]^S$ such that $f_{\lambda'}(a)$ converges to $f(a)$ for each $a \in S$ and this proves our claim.
Next, we claim that there exists an ideal I such that $\| a + I_{\lambda'} \|$ converges to $\| a + I \|$. Let p be a function on A defined by $p(a) = \lim_{\lambda'} \| a + I_{\lambda'} \|$. Clearly p is a semi-norm on A. Let $I = \{a \in A | p(a) = 0\}$ which is a closed two-sided ideal of A. Since $a + I \mapsto p(a)$ is a C^* norm on A/I and recall that C^* norm is unique, so $p(a) = \| a + I \|$. The result follows. □

Proposition 2.1.4. Let A be a C^*-algebra. The topology τ_s is stronger than the τ_w topology.

Proof. Let I_λ be a net in $\mathrm{Id}(A)$ and $I \in \mathrm{Id}(A)$. It is sufficient to show $I_\lambda \to I$ in τ_w whenever $\mathrm{I}_\lambda \to I$ in τ_s. Suppose $I_\lambda \to I$ in τ_s. Let $\mathcal{U} = \mathcal{U}(\{J_1, J_2, \ldots, J_n\})$ be a τ_w neighborhood of I. For each $i = 1, 2, \ldots, n$, there exists $a_i \in J_i \backslash I$. We have $\| a_i + I_\lambda \| \to \| a_i + I \| > 0$ Hence, there exists λ_0 such that $\| a_i + I_\lambda \| > 0$ for all i and all $\lambda \succeq \lambda_0$, i.e. $I_\lambda \in \mathcal{U}$ whenever $\lambda \succeq \lambda_0$. □

Proposition 2.1.5. Let A be a C^*-algebra. Then the closure of $\mathrm{Prim}(A)$ in τ_w topology is $\mathrm{Primal}(A)$.

Proof. Firstly, we show that the closure $\overline{\mathrm{Prim}(A)} \subseteq \mathrm{Primal}(A)$ by contradiction. Suppose there exists $I \in \overline{\mathrm{Prim}(A)} \backslash \mathrm{Primal}(A)$. Then there exists ideals $J_1, J_2, \ldots, J_n$ such that $J_1 J_2 \cdots J_n = 0$ but $J_i \not\subseteq I$ for all i. Let $\mathcal{U} = \mathcal{U}(\{J_1, J_2, \ldots, J_n\})$ which is an τ_w open neighborhood of I. Therefore $\mathcal{U} \cap \mathrm{Prim}(A)$ is non-empty. Choose a primitive ideal $P \in \mathcal{U}$. Since $J_1 J_2 \cdots J_n = 0 \subseteq P$, $J_i \subseteq P$ for some i. This contradicts to the fact that $P \in \mathcal{U}$.
Then we show that $\overline{\mathrm{Prim}(A)} \supseteq \mathrm{Primal}(A)$. Let $I \in \mathrm{Primal}(A)$. Let $\mathcal{U}$ be an arbitrary neighborhood of I. Then there exists $\mathcal{U}_1 = \mathcal{U}(\{J_1, J_2, \ldots, J_n\})$ such that

$I \in \mathcal{U}_1 \subseteq \mathcal{U}$. As I is primal , so $J_1J_2\cdots J_n \neq 0$. We may choose a primitive ideal P such that $J_1J_2\cdots J_n \nsubseteq P$. Then $P \in \mathcal{U}_1$, i.e. , $\mathcal{U}_1 \cap \mathrm{Prim}(A)$ is non-empty. □

Proposition 2.1.6. Let A be a C^*-algebra and $I \in \mathrm{Primal}(A)$. Let P_λ be a net of primitive ideals. Then $P_\lambda \to I$ in τ_w topology if and only if $P_\lambda \to P$ in the τ_w topology for any primitive ideal P containing I.

Proof. Suppose $P_\lambda \to I$. Let P be a primitive ideal containing I. Let $\mathcal{U} = \mathcal{U}(\{J_1, J_2, \ldots, J_n\})$ be an open neighborhood of P. It is obviously that $\mathcal{U}$ is also an open neighborhood of I. Hence $P_\lambda \in \mathcal{U}$ for sufficiently large λ, i.e., $P_\lambda \to P$. Conversely, suppose $P_\lambda \to P$ for all primitive ideals P containing I. Let $\mathcal{U}$ be an open neighborhood of I. Without loss of generality, we may assume that $\mathcal{U} = \mathcal{U}(\{J_1, J_2, \ldots, J_n\})$. For each i, since $J_i \nsubseteq I$, we can find a primitive ideal P_i such that $I \subseteq P_i$ but $J_i \nsubseteq P_i$ (If not, say, every P containing I also contains J_i, then $I = \cap_{P \supseteq I} P \supseteq J_i$. Contradiction). Define $\mathcal{U}_i = \mathcal{U}(\{J_i\})$ which is an open neighborhood of P_i. So, $P_\lambda \in \mathcal{U}_i$ eventually. Hence there exists λ_0 such that $P_\lambda \in \mathcal{U}_i \quad \forall \lambda \succeq \lambda_0$, $\forall i = 1, 2, \ldots, n$. In other word, $P_\lambda \in \mathcal{U}\ \forall \lambda \succeq \lambda_0$, therefore, $P_\lambda \to I$. □

Remark:

The above proposition reveal that in general Primal(A) is highly non-Hausdorff in the τ_w topology.

Also note that the only τ_w open neighborhood of the primal ideal A in Primal(A) is Primal(A) itself, therefore, Primal(A)is always τ_w compact but never τ_w Hausdorff. In order to truely reflect the topological properties of primal ideals, we consider Primal$'(A)$ instead, where Primal$'(A)$=Primal$(A)\backslash\{A\}$.

Proposition 2.1.7. Let A be a C^*-algebra, then:

(a) Primal$'(A)$ is τ_w compact if and only if Prim(A) is compact in the Jacobson topology.

(b) The following statments are equivalent:

(b1) Primal$'(A)$ is Hausdorff.

(b2) Prim(A) is Hausdorff in hull-kernel topology.

(b3) Primal$'(A)$=Prim(A) and any primitive ideal is maximal.

Proof. (a) Suppose Primal$'(A)$ is τ_w compact. Let $\{P_\lambda\}$ be a net in Prim(A). Regarding $\{P_\lambda\}$ as a net in Primal$'(A)$, we can find a subnet $\{P_{\lambda'}\}$ which τ_w converges to some $I \in$ Primal$'(A)$. Since $I \neq A$, we can find a primitive ideal $P \supseteq I$. By proposition 2.1.6, $\{P_{\lambda'}\} \to P$ in τ_w topology. But the τ_w topology coincides with the hull-kernel topology in Prim(A), hence Prim(A) is compact. Conversely, suppose Prim(A) is compact in the hull-kernel topology. Let $\{I_\lambda\}$ be a net in Primal$'(A)$. For each λ, we choose a primitive ideal P_λ such that $P_\lambda \supseteq I_\lambda$. Then there exits a subnet $\{P_{\lambda'}\}$ of $\{P_\lambda\}$ which converges to some primitive ideal P. By looking at the τ_w open neighborhoods of P, we immediately conclude that $I_{\lambda'} \to P$ in the τ_w topology.

(b) (b1)$\Rightarrow$(b2) is trival since Prim(A) is just a subspace of Primal$'(A)$ in τ_w topology.

(b2)$\Rightarrow$(b3):

Firstly, we show that Primal$'(A)$=Prim(A). Suppose not, then there exists $I \in$ Primal$'(A)\backslash$Prim(A). Since $I \neq A$, hull(I) is non-empty. (hull(I)=$\{P \in$ Prim$(A)|\ I \subseteq P\}$) Also note that hull(I) cannot be a singleton (If hull(I)=$\{P\}$, then I=ker hull(I)=P, contradiction !). Choose distinct $P_1, P_2 \in$hull(I). By proposition 2.1.5, there exists a net $\{P_\lambda\}$of primitive ideals which converges to I in τ_w topology. By proposition 2.1.6, $\{P_\lambda\}$ will converge to P_1, P_2 simultaneously. Contradiction.

Secondly, we show that each primitive ideal is maximal. If not, say, there exists a primitive P_1 which in not maximal, then we can find a primitive ideal P_2 such that $P_1 \subset P_2$. Take $P_\lambda \equiv P_1$. Clearly $\{P_\lambda\}$ converges to both P_1 and P_2. Contradiction !

(b3)$\Rightarrow$(b2):

Refer to [6] [p.63]

(b3)$\Rightarrow$(b1):

(b3)$\Rightarrow$(b2) and (b3)+(b2)$\Rightarrow$(b1). $\square$

Now we generalize some of the classical topological properties of Prim(A) to Primal(A).

Theorem 2.1.8. Let A be a C^*-algebra. Then

(a) Primal(A) and Primal$'(A)$ are Baire spaces in τ_w topology.

(b) Suppose further that A is separable, then $(\text{Primal}(A), \tau_w)$ is second countable.

Proof. (a) We recall that Prim(A) is a Baire space in the hull-kernel topology [2] [Corollary 3.4.13]. Let $\{U_n\}$ be countably many open dense subsets of Primal(A) (in τ_w topology). Note that $U_n \cap \text{Prim}(A)$ is a dense open subset of Prim(A), so $(\cap_{n=1}^{\infty} U_n) \cap \text{Prim}(A) = \cap_{n=1}^{\infty}(U_n \cap \text{Prim}(A))$ which is dense in Prim(A). Therefore $\cap_{n=1}^{\infty} U_n$ is dense in Primal(A). Hence Primal(A) is a Baire space. The same argument apply to Primal$'(A)$.

(b) Suppose A is separable, then Prim(A) is second countable. [2] [Proposition 3.3.4]. Let $\{U_n\}$ be a countable base of Prim(A) in the hull-kernel topology. Fix a proper primal ideal I_0. Let V be an arbitrary open τ_w neighborhood of I_0. Choose $\mathcal{U} = \mathcal{U}(\{J_1, J_2, \ldots, J_n\})$ such that $I_0 \in \mathcal{U} \subseteq V$. For each i, there exists a primitive ideal P_i such that $I_0 \subseteq P_i$ and $J_i \nsubseteq P_i$. Then choose k_i such that $P_i \in U_{k_i}$. Lastly, define $V' = V'(k_1, k_2, \ldots, k_n) = \{I \in \text{Primal}(A) \mid \text{For each } i,\ U_{k_i} \text{ contains an (primitive) ideal which contains } I\}$. We assert that V' is τ_w open and $I_0 \in V' \subseteq \mathcal{U} \subseteq V$. Then $\{V'(k_1, k_2, \ldots, k_n) \mid n = 1, 2, \ldots; k_1, k_2, \ldots, k_n \in \mathbb{N}\}$ constitutes a countable base of τ_w. Clearly $I_0 \in V'$, we need to show that V' is open. For each i, there exists an ideal K_{k_i} such that $U_{k_i} = \{P \in \text{Prim}(A) \mid K_{k_i} \not\subseteq P\}$. Now $V'^c = \{I \mid \text{There exists } i, U_{k_i} \cap \text{hull}(I) = \phi\}$. But $U_{k_i} \cap \text{hull}(I) = \phi \Leftrightarrow \text{hull}(I) \setminus \text{hull}(K_{k_i}) = \phi \Leftrightarrow \text{hull}(I) \subseteq \text{hull}(K_{k_i}) \Leftrightarrow K_{k_i} \subseteq I$. So V'^c is just the finite unions of the τ_w closed sets $\{I \mid K_{k_i} \subseteq I\}$. □

Theorem 2.1.9. Let A be a C^*-algebra and $a \in A$.

(a) For each $\varepsilon > 0$, the set $\{I \in \text{Primal}(A) \mid \| a + I \| \geq \varepsilon\}$ is compact in τ_w topology.

(b) For each $\varepsilon \geq 0$, there exists an ideal J such that $\{I \in \text{Primal}(A) \mid \| a + I \| > \varepsilon\} = \{I \in \text{Primal}(A) \mid J \not\subseteq I\}$. Hence the function $I \rightarrow \| a + I \|$ is lower semi-continuous in the τ_w topology.

Proof. (a)Denote the set $\{I \in \text{Primal}(A) \mid \| a + I \| \geq \varepsilon\}$ by B. Let $\{U_\lambda\}$ be an (τ_w) open covering of B. Then $\{U_\lambda \cap \text{Prim}(A)\}$ is an opening covering of the compact set $B \cap \text{Prim}(A)$. We can find $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $B \cap \text{Prim}(A) \subseteq \cup_{i=1}^{i=n}(U_{\lambda_i} \cap \text{Prim}(A))$. Let $I \in B$. Choose a primitive ideal P such that $I \subseteq P$ and $\| a + I \| = \| a + P \|$. Then $P \in B \cap \text{Prim}(A)$, so, $P \in U_{\lambda_i}$ for some i and hence $I \in U_{\lambda_i}$. Therefore $\{U_{\lambda_i}\}_{i=1}^{i=n}$ covers B, i.e. B is (τ_w) compact.

(b) We define $X = \{I \in \text{Primal}(A) \mid \| a + I \| \leq \varepsilon\}$ and $J = \ker(X \cap \text{Prim}(A))$. We need to show $X = \{I \in \text{Primal}(A) \mid J \subseteq I\}$.

Let $I \in X$. Let P be an arbitrary ideal such that $I \subseteq P$, then $P \in X \cap \text{Prim}(A)$. Since $X \cap \text{Prim}(A)$ is closed in $\text{Prim}(A)$, so $J \subseteq P$. Lastly, $J \subseteq \cap P = I$ (where the intersection is taken over all the primitive ideals P containing I).

Conversely, let I be a primal ideal such that $J \subseteq I$. Suppose $I \notin X$. Choose a primitive ideal P such that $I \subseteq P$ and $\| a + P \| = \| a + I \| > \varepsilon$. So $P \notin X \cap \text{Prim}(A)$. As $X \cap \text{Prim}(A)$ is closed in the Jacobson topology, i.e. $J \not\subseteq P$, contradiction ! □

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Now, we generalize the classical result about pure states and primitive ideals to a result about factorial states and primal ideals.

Recall that a factorial state ϕ of a C^*-algebra A is a state such that the associated representation (π_ϕ, H) is a factor representation, (i.e., the von-Neumann algebra generated by $\pi_\phi(A)$ is a factor). Archbold proved that ϕ is a weak* limit of factorial states if and only if $\ker(\pi_\phi)$ is a primal ideal. [6] [Theorem 3.5]. Later, he also discovered the topological relation between these two objects.

Lemma 2.1.10. Let A be a C^*-algebra. We denote
$\mathrm{QS}(A) = \{\phi \mid \phi$ is a positive linear functional on A with norm $\leq 1\}$.
$\mathrm{S}(A) =$ the set of all states of A.
$\mathrm{PS}(A) =$ the set of all pure states of A.
$\mathrm{F}(A) =$ the set of all factorial states of A.
Given a positive linear functional ϕ, we denote the associated cyclic representation arised form GNS construction by π_ϕ.
Define a map $\theta : \mathrm{QA}(A) \to \mathrm{Id}(A)$ by $\theta(\phi) = \ker(\pi_\phi)$. The map θ is continuous if $\mathrm{QA}(A)$ and $\mathrm{Id}(A)$ are endowed with the weak* and the τ_w topologies respectively.

Proof. Let $\{\phi_\lambda\}$ be a net in $\mathrm{QA}(A)$ which converges to ϕ in weak* topology. Denote $I = \theta(\phi)$. Let $\mathcal{U}_1$ be a τ_w neighborhood of I. Choose ideals $J_1, J_2, \ldots, J_n$ such that $I \in \mathcal{U} := \mathcal{U}(J_1, J_2, \ldots, J_n) \subseteq \mathcal{U}_1$. Denote $I_\lambda = \theta(\phi_\lambda)$. We prove by contradiction, suppose I_λ is not eventually in $\mathcal{U}$. We obtain a subnet $\{\phi_{\lambda'}\}$ such that $I_{\lambda'} \notin \mathcal{U}$. We get some k and a subnet of $\{\phi_{\lambda'}\}$ (still denoted by $\{\phi_{\lambda'}\}$) such that $J_k \subseteq I_{\lambda'}$ for all λ'. Now choose $x_0 \in J_k \backslash I$. Then $\phi(x_0^* x_0) \neq 0$ or $\phi(x_0 x_0^*) \neq 0$. Without loss of generality, assume $\phi(x_0^* x_0) \neq 0$, then $\phi_{\lambda'}(x_0^* x_0) \neq 0$ for sufficient large λ'. Hence $x_0 \notin I_{\lambda'}$ (for sufficient large λ'). Contradiction. □

Theorem 2.1.11. Let A be a C^*-algebra and let $S = \theta^{-1}(\mathrm{Primal}(A))$. We endow $\mathrm{QS}(A)$ and $\mathrm{Id}(A)$ with the weak* and the τ_w topologies. Then:

(i) S is weak* compact and the $\overline{\mathrm{F}(A)} \subseteq S$.

(ii) $S = \{\alpha\phi \mid \alpha \in [0,1] \text{ and } \phi \in \overline{\mathrm{F}(A)} \cap \mathrm{S}(A)\}$

(iii) $\theta\mid_S$ is continuous and $\theta\mid_S : S \to \theta(S)$ is open.

(iv) $\theta\mid_{\overline{\mathrm{F}(A)}\cap \mathrm{S}(A)}$ is continuous and $\theta\mid_{\overline{\mathrm{F}(A)}\cap \mathrm{S}(A)}: \overline{\mathrm{F}(A)} \cap \mathrm{S}(A) \to \theta(\overline{\mathrm{F}(A)} \cap \mathrm{S}(A))$ is open.

Proof. (i) $\mathrm{Primal}(A) = \overline{\mathrm{Prim}(A)}$ is closed and θ is continuous , so S is closed. Recall that $\mathrm{QA}(A)$ is compact, so S is compact (in the weak* topology).

If $\phi \in \mathrm{F}(A)$, then $\theta(\phi)$ is a prime hence a primal ideal. So $\mathrm{F}(A) \subseteq S$ and hence $\overline{\mathrm{F}(A)} \subseteq \overline{S} = S$.

(ii) It is obvious that $S \subseteq \{\alpha\phi \mid \alpha \in [0,1] \text{ and } \phi \in \overline{\mathrm{F}(A)} \cap \mathrm{S}(A)\}$.

Coversely, if $\alpha \in [0,1]$ and $\phi \in \overline{\mathrm{F}(A)} \cap \mathrm{S}(A)$. We note that $\theta(\alpha\phi) = \theta(\phi)$ if $\alpha \neq 0$ and $\theta(\alpha\phi) = A$ if $\alpha = 0$. Clearly A and $\theta(\phi)$ are primal ideals.

(iii)The continuity of $\theta\mid_S$ follows from lemma. We now show the openness.

Let $\phi \in S$. Let $x_1, x_2, \ldots, x_m \in A$ and $\varepsilon > 0$. Define $U := \{\phi' \in S \mid \mid \phi(x_i) - \phi'(x_i) \mid < \varepsilon \text{ for all } i = 1, 2, \ldots, m\}$. Note that such U constitutes a local base of ϕ, so it is sufficient to show that $\theta(U)$ is a neighborhood of $\theta(\phi)$. Recall that we can identify a positive linear functional φ on A which satisfies $I \subseteq \ker(\pi_\varphi)$ with a positive linear functional φ' on A/I via $\varphi'(x+I) = \varphi(x)$, $x \in A$, moreover, $\parallel \varphi \parallel = \parallel \varphi' \parallel$ and φ is pure iff φ' is pure. Let $I = \theta(\phi)$ which is primal. By regarding ϕ as a state on A/I, we can approximate (in weak* topology) ϕ by positive linear functionals of the form $\sum_{i=1}^n \alpha_i\phi_i$. where $\alpha_i \in (0,1)$, $\sum_{i=1}^n \alpha_i \leq 1$, and ϕ_i are pure states on A/I. Choose suitable α_i and ϕ_i and define $\psi = \sum_{i=1}^n \alpha_i\phi_i$ such that $\mid \psi(x_j) - \phi(x_j) \mid < \varepsilon/4$ for all $j = 1, 2, \ldots, m$. (Here we identify ϕ_i as pure states on A with $I \subseteq \ker(\pi_{\phi_i})$).Define a weak* open neighborhood of 0, $W := \{\psi' \in A^* \mid \mid \psi'(x_j) \mid < \varepsilon/4, \text{for all } j = 1, 2, \ldots, m\}$. Clearly $(\psi + W) \cap S \subseteq U$.

By [2] [3.4.11],for each i, $\theta((\phi_i+W)\cap \mathrm{PS}(A))$ is an open set in $\mathrm{Prim}(A)$, so there exists an ideal J_i such that $\theta((\phi_i+W)\cap \mathrm{PS}(A)) = \{P \in \mathrm{Prim}(A) \mid J_i \not\subseteq P\}$. Define an open set in $\mathrm{Primal}(A)$ by $V := \{J \in \mathrm{Primal}(A) \mid J_i \not\subseteq J$ for all $i = 1, 2, \ldots, n\}$. We claim that $I \in V \cap \theta(S) \subseteq \theta(U)$ and finish our proof.

Note that $\phi_i(J_i) \neq \{0\}$, so $J_i \not\subseteq I$ for all i (as ϕ_i vanishes on I) and hence $I \in V \cap \theta(S)$.

To show $V\cap\theta(S) \subseteq \theta(U)$, we let $J \in V\cap\theta(S)$. For each i, there exists a primitive ideal P_i such that $J \subseteq P_i$ but $J_i \not\subseteq P_i$. As $P_i \in \theta((\phi_i+W)\cap \mathrm{PS}(A))$, there exists a pure state $\sigma_i \in \phi_i + W$ such that $\theta(\sigma_i) = P_i$. Define $\sigma = \sum_{i=1}^{n} \alpha_i\sigma_i$, it can be verified easily that $\sigma \in \psi + W$. Let $\rho \in S$ be arbitrary such that $\theta(\rho) = J$. For each $k \in \mathbb{N}$, we define $\rho_k = \frac{1}{k}\rho+(1-\frac{1}{k})\sigma$ and claim: $J = \theta(\rho_k)$ and $\rho_k \in \psi+W\cap S$ for sufficient large k.

Let $x \in J$. Note that $J \subseteq P_i$, so $\sigma_i(x^*x) = 0$ for all i and hence $\sigma(x^*x) = 0$. Also $\rho(x^*x) = 0$, so $\rho_k(x^*x) = 0$. Similarly, $\rho_k(xx^*) = 0$ and hence $x \in \theta(\rho_k)$.

Conversely, if $x \in \theta(\rho_k)$, then $\rho_k(x^*x) = \rho_k(xx^*) = 0$, so $\rho(x^*x) = \rho(xx^*) = 0$ and hence $s \in J$. This shows $J = \theta(\rho_k)$.

Note that $\rho_k \in S$ and $\rho_k \to \sigma \in \psi + W$. By the openness of $\psi + W$, $\rho_k \in \psi + W$ for sufficiently large k.

(iv) The continuity follows form Lemma 1.10. The openness follows from (iii). For, let U be an open subset of $\overline{\mathrm{F}(A)} \cap \mathrm{S}(A)$. By (ii), there exists an open subset V of S such that $U = V \cap (\overline{\mathrm{F}(A)} \cap \mathrm{S}(A))$. By (iii) $\theta(V)$ is open in $\theta(S)$. If $0\notin V$, then $\theta(U) = \theta(V)$ and hence $\theta(U) = \theta(V) \cap \theta(\overline{\mathrm{F}(A)} \cap \mathrm{S}(A))$ which is open in $\theta(\overline{\mathrm{F}(A)} \cap \mathrm{S}(A))$. If $0\in V$, then $\theta(V) = \theta(U) \cup \{A\}$, we also have $\theta(U) = \theta(V) \cap \theta(\overline{\mathrm{F}(A)} \cap S(A))$. □

Theorem 2.1.12. Let A be a C^*-algebra. Then both $\mathrm{Primal}(A)$ and $\mathrm{Primal}'(A)$ are locally compact in the τ_w topology. The term "locally compact" here is of the following sense: A topological space X is said to be locally compact if for each

$x \in X$, and each neighborhood U of x, there exists a compact neighborhood K of x such that $K \subseteq U$.

Proof. Let $I \in \text{Primal}'(A)$ and W be an open neighborhood of I in $\text{Primal}'(A)$, there exists an open neighborhood $\mathcal{U} = \mathcal{U}(\{J_1, J_2, \ldots, J_n\}) \cap \text{Primal}'(A)$ such that $I \in \mathcal{U} \subseteq W$. For each i, choose a primitive ideal P_i such that $I \subseteq P_i$ but $J_i \not\subseteq P_i$. Let $Q = \bigcap_{i=1}^{n} P_i$ then $I \subseteq Q$ and $Q \in \mathcal{U}$. (Since I is primal, Q is primal too.) For each i, choose a pure state ϕ_i on A such that $\theta(\phi_i) = P_i$. (where θ is the map defined in Lemma 2.1.10). Define a state $\phi = 1/n \sum_{i=1}^{n} \phi_i$ and note that $\theta(\phi) = Q$, so $\phi \in S$. By the continuity of $\theta\mid_S$, $\theta\mid_S^{-1}(\mathcal{U})$ is an open neighborhood of ϕ in S. As S is compact and Hausdorff (Theorem 2.1.11 (i)), there exists an open neighborhood of ϕ in S whose closure (in S) is contained in $\theta\mid_S^{-1}(\mathcal{U})$. Denote the closure (in S) of such open neighborhood by M. By Theorem 2.1.11 (iii), $V := \theta(M)$ is a compact neighborhood of Q is $\theta(S)$. We have I$\subseteq Q \in V \subseteq \mathcal{U}$. However, V may not be compact in $\text{Primal}'(A)$, we need to modify V and obtain a compact neighborhood of I. We define $N = \{J \in \text{Primal}'(A) \mid J \subseteq K \text{ for some } K \in V\}$ and show that N is a compact neighborhood of Q contained in $\mathcal{U}$. Clearly $V \subseteq N \subseteq \mathcal{U}$. Let $\{Y_\lambda\} \subseteq \text{Primal}'(A)$ be an open covering of N. Then $\{Y_\lambda \cap \theta(S)\}$ is an open covering of V in $\theta(S)$. By compactness of V, we can choose a finite subcover, say $V \subset \theta(S) \cap (Y_{\lambda_1} \cup Y_{\lambda_2} \cup \cdots \cup Y_{\lambda_k})$. Now if $J \in N$, then $J \subseteq K$ for some $K \in V$, so $K \in Y_{\lambda_i}$ and hence $J \in Y_{\lambda_i}$. So $\{Y_{\lambda_i}\}$ covers N. This shows the compactness of N.

Since V is a neighborhood of Q in $\theta(S)$, there exits an open set Z in $\text{Primal}(A)$ such that $Q \in Z \cap \theta(S) \subseteq V$. Clearly $A \notin Z$. If $J \in Z$, we use the same technique in constructing Q to obtain an ideal Q' such that $J \subset Q'$ and $Q' \in Z \cap \theta(S)$. So $Q' \in V$ and $J \in N$, this shows $Z \subseteq N$.Hence N is a neighborhood of Q in $\text{Primal}'(A)$ and hence a neighborhood of I. This proves the local compactness of $\text{Primal}'(A)$.

Then we prove the local compactness of Primal(A). Let $I \in$ Primal(A) and W a neighborhood of I in Primal(A). If $I = A$, we have $W =$ Primal(A) which is already compact. Suppose $I \neq A$, then there exists a compact set V such that $I \in V \subseteq W \cap$ Primal$'(A)$ and V is a neighborhood of I in Primal$'(A)$. Since Primal$'(A)$ is open in Primal(A), V is also a neighborhood of I in Primal(A) $\square$

Proposition 2.1.13. Let A be a C^*-algebra. Then

(a) Primal(A) is compact in the τ_s topology.

(b) The following conditions are equivalent:

(i) Primal$'(A)$ is τ_s compact.

(ii) Prim(A) is compact in the hull-kernel topology.

(iii) $A \notin \overline{\text{Prim}(A)}$, where $\overline{\text{Prim}(A)}$ denotes the τ_s closure of Prim(A).

(iv) A is an isolated point in Primal(A) with respect to the τ_s topology.

Remark: If A is unital, all the conditions (i),(ii),(iii),(iv) hold.

Proof. (a)By Proposition 2.1.4 and Proposition 2.1.5, Primal(A) is τ_w closed and hence τ_s closed. By Proposition 2.1.3, Primal(A) is τ_s compact.

(b) (i)$\Rightarrow$ (ii). Suppose Primal$'(A)$ is τ_s compact. Let $\{P_\lambda\}$ be a net in Prim(A). By regarding it as a net of primal ideals, there exists a subset $\{P_{\lambda'}\}$ and a primal ideal $I \in$ Primal$'(A)$ such that $P_{\lambda'} \to I$ in τ_s topology and hence $P_{\lambda'} \to I$ in τ_w topology (since τ_s is finer than τ_w). As $I \neq A$, there exists a primitive ideal P such that $I \subseteq P$. By proposition 2.1.6, $P_{\lambda'} \to P$ in the hull-kernel topology.

(ii)$\Rightarrow$(iii) Suppose Prim(A) is compact in the hull-kernel topology. If A is in the τ_s closure of Prim(A), then there exists a net of primitive ideals $\{P_\lambda\}$ which converges τ_s to A. On the other hand, there exists a subnet $\{P_{\lambda'}\}$ and a primitive ideal P such that $P_{\lambda'} \to P$ in the hull-kernel topology. Choose $x \in A \backslash P$. By Dixmier's result, we have $0 < \| x + P \| \leq \liminf_{\lambda'} \| x + P_{\lambda'} \|$ This contradicts to $\lim_{\lambda'} \| x + P_{\lambda'} \| = \| x + A \| = 0$.

(iii)$\Rightarrow$(iv) Suppose A in not an isolated point in Primal(A) with respect to the τ_s topology. Then there exists a net $\{I_\lambda\} \subseteq \text{Primal}'(A)$ which τ_s converges to A. For each λ, choose a primitive ideal P_λ such that $I_\lambda \subseteq P_\lambda$. Now for each $x \in A$, $\| x + P_\lambda \| \leq \| x + I_\lambda \| \rightarrow \| x + A \| = 0$. Hence $P_\lambda \rightarrow A$ in τ_s topology, contradiction.

(iv)$\Rightarrow$(i) By (iv), $\{A\}$ is an τ_s open set, so $\text{Primal}'(A) = \text{Primal}(A) \backslash \{A\}$ is τ_s closed and hence compact. □

2.2 Minimal Primal Ideals and Their Topology

Definition 2.2.1. A primal ideal I of a C^*-algebra A is said to be minimal if for each primal ideal J which satisfies $J \subseteq I$, then $J = I$. We denote the set of all minimal primal ideals of A by min-Primal(A).

Remark:

By an application of Zorn's Lemma, it can be proved that each primal ideal contains a minimal primal ideal. Also, if Prim(A) is finite, then min-Primal(A)=min-Prim(A).We prove the last statement as follow.

Let I be a primal ideal. Define $\mathcal{C} = \{J \in \text{Primal}(A) \mid J \subseteq I\}$ which is non-empty and is partially ordered by $\supseteq$. Let $\{J_\lambda\}$ be a chain in $\mathcal{C}$. Define $J = \cap_\lambda J_\lambda$. We claim that J is a primal ideal. If not, then there exists ideals I_1,I_2,..., I_n such that $I_1 I_2 \cdots I_n = 0$ but $I_i \nsubseteq J$ for each i. For each i, there exists λ_i such that $I_i \nsubseteq J_{\lambda_i}$. Among J_{λ_1}, J_{λ_2},..., J_{λ_n}. We may choose a "smallest" one, say J_{λ_i} such that $J_{\lambda_i} \subseteq J_{\lambda_j}$ for all $j = 1, 2, \ldots, n$. Then I_1,I_2,...,I_n all are not contained in J_{λ_i}. Contradiction ! So $J \in \mathcal{C}$ which is an upper-bound of the chain $\{J_\lambda\}$. By Zorn's Lemma, $\mathcal{C}$ has a maximal element.

Let I be a minimal primal ideal. Then there exists a net of primitive ideal $\{P_\lambda\}$ such that $P_\lambda \rightarrow I$ in the τ_w topology (Proposition 2.1.5). Let Q be an arbitrary primitive ideal containing I, then $P_\lambda \rightarrow Q$. Since Prim(A) is finite, there exists

a primitive ideal R such that P_λ is frequently equal to R. Hence $R \subseteq Q$. Taking intersection over all such Q, we have $R \subseteq I$. As R is primal and I is minimal primal, $R = I$. This shows min-Primal(A)$\subseteq$ min-Prim(A).
The converse is easy.

Theorem 2.2.2. Let A be a C^*-algebra and $\iota : (\text{Primal}(A), \tau_w) \to (\text{Primal}(A), \tau_s)$ be the identity map. Let $I \in$ Pirmal(A), then ι is continuous at I if and only if I is a minimal primal ideal.

Proof. Suppose ι is continuous at I but I is not minimal primal. We choose a primal ideal J such that $J \subset I$. Define a constant net $J_\lambda \equiv J$. Clearly $J_\lambda \to I$ in τ_w topology but not in τ_s topology. Contradiction. This proves the "only if" part. Conversely, suppose I is minimal primal ideal. Let $\{I_\lambda\} \in$ Primal(A) be a net such that $I_\lambda \to I$ in τ_w topology. Let $\{I_\alpha\}$ be an arbitrary subnet of $\{I_\lambda\}$. By compactness of Primal(A) in Id(A) with respect to τ_s, there exists a subnet $\{I_\beta\}$ of $\{I_\alpha\}$ and a primal ideal J such that $I_\beta \to J$ in τ_s topology. Now for each $x \in A$, $\| x + J \| = \lim_\beta \| x + \mathrm{I}_\beta \| \geq \| x + I \|$. (Where the last inequality follows from Proposition 2.1.9(b).) So $J \subseteq I$. By minimality of I, we have $J = I$, and hence every subnet of $\{I_\lambda\}$ has a subnet which converges to I in the τ_s topology. Therefore, $I_\lambda \to I$ (with respect to τ_s). This shows the continuity of the map ι at I and proves the "if" part. □

Corollary 2.2.3. Let A be a C^*-algebra, then:
(i) The two topologies τ_s and τ_w coincide on min-Primal(A).
(ii) min-Primal(A) $\subseteq \overline{\text{Prim}(A)}$(w.r.t.$\tau_s$) $\subseteq$ Primal(A).
(iii) Let $\theta : \overline{\text{F}(A)} \cap \text{S}(A) \longrightarrow \text{Primal}'(A)$ defined by $\theta(\phi) = \ker(\pi_\phi)$. Equip $\overline{\text{F}(A)} \cap \text{S}(A)$ and Primal$'(A)$ with the weak* and the τ_s topology, then θ is continuous at ϕ if and only if $\theta(\phi)$ is a minimal primal ideal.

Proof. (i) Follows directly from Theorem 2.2.2

(ii) Let $I \in$ min-Primal(A). By proposition 2.1.5, there exists a net $\{P_\lambda\}$ of primitive ideals such that $P_\lambda \to I$ in τ_w topology and hence in τ_s topology (Theorem 2.2.2). Therefore $I \in \overline{\text{Prim}(A)}$, the τ_s closure of Prim(A). The other inclusion is obvious because τ_s closure of Prim$(A) \subseteq \tau_w$ closure of Prim(A) = Primal(A).

(iii) Let $\phi \in \overline{\mathrm{F}(A)} \cap \mathrm{S}(A)$ such that $\theta(\phi)$ is a minimal primal ideal. Let $\{\phi_\lambda\}$ be an arbitrary net in $\overline{\mathrm{F}(A)} \cap \mathrm{S}(A)$ which converges to ϕ. By theorem 2.1.1(iv), $\theta(\phi_\lambda) \to \theta(\phi)$ in τ_w and hence in τ_s topology by minimality of $\theta(\phi)$ and Theorem 2.2.2. This proves the continuity of θ at ϕ.

Conversely, suppose θ is continuous at ϕ. Assume $\theta(\phi)$ is not minimal, then there exists a primal ideal J such that $J \subset \theta(\phi)$. Denote $I := \theta(\phi)$. Choose a primitive ideal P such that $J \subseteq P$ but $I \nsubseteq P$. Choose a pure state ψ such that P=ker(π_ψ). For each positive integer n, define a state $\phi_n := \frac{1}{n}\psi + (1 - \frac{1}{n})\phi$. Then ker$(\pi_{\phi_n})$= ker$(\pi_\phi)\cap$ker$(\pi_\psi)$=$I \cap P \supseteq J$. Since J is primal, ker(π_{ϕ_n}) is also primal and hence $\phi_n \in \overline{\mathrm{F}(A)} \cap \mathrm{S}(A)$. Now $\phi_n \to \phi$, so $I \cap P = \theta(\phi_n) \to \theta(\phi) = I$ in τ_s topology. This is a contradiction ! $\square$

Now, we investigate the relation between separated points and minimal primal ideals. Suppose A is a C^*-algebra and $P \in$ Prim(A). P is said to be a separated point if for any primitive ideal Q, with $P \nsubseteq Q$, there exists disjoint open neighborhoods for P and Q.

Remark

(i) If P is a separated point, then it must be minimal primitive. If not, suppose there exists a primitive ideal P' such that $P' \subset P$. Note that every open neighborhood of P' must also be an open neighborhood of P. Contradiction.

(ii) (Dixmier's Result) P is a separated point if and only if for each $a \in A$, the map $R \mapsto \| a + R \|$ defined on Prim(A) is continuous at P with respect to the hull-kernel topology.

Proposition 2.2.4. Let A be a C^*-algebra. Let P be a primitive ideal. Then the following propositions are equivalent:

(i) P is a minimal primal ideal.

(ii) P is a separated point.

Proof. (i)$\Rightarrow$(ii). Suppose P is a minimal primal ideal. Let $a \in A$ be arbitrary. By remark (ii), it is sufficient to show the continuity of the map $R \mapsto \| a + R \|$ at P. Let $\{P_\lambda\}$ be a net which converges to P. By minimality of P and Theorem 2.2.2, $\{P_\lambda\} \to P$ in the τ_s topology and hence map $R \mapsto \| a + R \|$ is continuous at P.

(ii)$\Rightarrow$(i). Suppose P is a separated point but P is not a minimal primal ideal. We can find a primal ideal I such that $I \subset P$. Choose a primitive ideal Q such that $I \subseteq Q$ and $P \nsubseteq Q$. By Proposition 2.1.5, there exists a net $\{P_\lambda\}$ of primitive ideals such that $P_\lambda \to I$ in τ_w topology, hence $\{P_\lambda\}$ converges to both P and Q simultaneously. (Proposition 2.1.6). So P is not separated, contradiction. $\square$

Theorem 2.2.5. Let A be a C^*-algebra. Then min-Primal(A) is a Baire Space.

Proof. Let $\{U_n\}$ be a countable collection of open sets in min-Primal(A), each of which is dense in min-Primal(A). Let V_n be an τ_w open set in Primal(A) such that U_n=$V_n \cap$ min-Primal(A). Note each V_n is τ_w dense in Primal(A). For, let J be a primal ideal and W a τ_w open neighborhood of J. Then $W \cap$min-Primal(A) is open and non-empty in min-Primal(A). By density of U_n, $U_n \cap W$ is non-empty and hence $V_n \cap W$ is non-empty too. Define $V := \cap_{n=1}^{\infty} V_n$ which is dense in Primal(A) by Proposition 2.1.8. Define $U := \cap_{n=1}^{\infty} U_n$. Let $I \in$ min-Primal(A) and W be an open neighborhood of I in min-Primal(A). Write $W = W' \cap$

min-Primal(A), where W' is a τ_w open set in Primal(A). Choose $J \in W' \cap V$ and let J_1 be a minimal primal ideal contained in J. Then $J_1 \in W'$. On the other hand, $J \in V_n$ implies $J_1 \in V_n$ for each n and so $J_1 \in V$. Hence $J_1 \in V \cap W' \cap$ min-Primal(A)=$U \cap W$. This shows the density of U in min-Primal(A). □

Proposition 2.2.6. Let A be a C^*-algebra. If $I \in$ min-Primal(A) is an isolated point, then I is an prime ideal.

Proof. By Proposition 2.1.5, there exists a net $\{P_\lambda\}$ of primitive ideals such that $P_\lambda \to I$ in the τ_w topology. For each λ, choose a minimal primal ideal J_λ contained in P_λ. Clearly $J_\lambda \to I$ in the τ_w topology. As I is isolated, $J_\lambda = I$ for sufficiently large λ. Without loss of generality, we may assume $J_\lambda = I$ for all λ. So $I \subseteq P_\lambda$. Let I_1 and I_2 be two ideals such that $I_1 I_2 \subseteq I$. For each λ, $I_1 \subseteq P_\lambda$ or $I_2 \subseteq P_\lambda$. Therefore, either $\{P_\lambda\}$ frequently contains I_1 or $\{P_\lambda\}$ frequently contains I_2. Without loss of generality, assume $\{P_\lambda\}$ frequently contains I_1. By minimality of I and Theorem 2.2.2, $P_\lambda \to I$ in the τ_s topology. Therefore, for each $x \in I_1$, $\| x + I \| = \lim_\lambda \| x + P_\lambda \| \leq \| x + I_1 \| = 0$. So $I_1 \subseteq I$. □

In Prim(A), hull-kernel process gives the Jacobson topology. It is natural to ask "On min-Primal(A), which kinds of topology will be produced by the hull-kernel process ?". In fact, the hull-kernel process produces the τ_s ($=\tau_w$ and we denote it by τ) topology if it really produces a topology.

Proposition 2.2.7. Let A be a C^*-algebra. Then the following conditions are equivalent:

(i) The hull-kernel process satisfies the Kuratowski's closure operator axioms.

(ii) For each subset E of min-Primal(A), the τ closure of E in min-Primal(A)=hull(ker(E)); where ker(E):=$\cap_{I \in E} I$ and for each ideal I, hull(I):=$\{J \in \text{min-Primal}(A) \mid I \subseteq J\}$.

Proof. (ii)$\Rightarrow$(i) is obvious. Suppose (i) holds. Let S be a subset of min-Primal(A). Clearly hull(ker(S)) is τ_w closed, hence is τ_s closed. So the τ_s closure of S is contained in hull(ker(S)). Conversely, suppose hull(ker(S)) is not contained in the τ_s closure of S. Choose I $\in$hull(ker(S)) such that I is not in the τ_s closure of S. There exists an open neighborhood of I disjoint from the τ_s closure of S. There exists ideals J_1, J_2,...,J_n of A such that $J_i \nsubseteq I$ and $\mathcal{U} = \mathcal{U}(\{J_1, J_2, \ldots, J_n\})$ has empty intersection with the τ_s closure of S. Hence $S \subseteq \overline{S} \subseteq (\mathcal{U}^c \cap \text{Min-Primal}(A)) = \cup_{i=1}^{n}\text{hull}(J)_i$. For each i, hull(J_i)=hull ker hull (J_i) which is a hull-kernel closed set and hence $\cup_{i=1}^{n}\text{hull}(J_i)_i$ =hull(K) for some ideal K. Now S $\subseteq$hull(K), and hence hull(K) $\supseteq$hull ker(S). As I $\notin$hull(K) so I $\notin$hull ker(S). Contradiction. $\square$

Chapter 3

Topological Representation of C^*-Algebras

3.1 Introduction

In this chapter, we will give an important application to primal ideals, namely, using min-Primal(A) as the base space and represents a C^*-algebra as an algebra of operator fields. Before starting the discussion, we need some preliminary, those materials can be found in J.M.G Fell. [3]

Definition 3.1.1. Let X be a locally compact Hausdorff space. For each $t \in X$, we associate a Banach space A_t. A vector field x is an element in $\prod_{t\in X} A_t$.
A continuity structure F is a subset of $\prod_{t\in X} A_t$ such that:
(i) F is a vector subspace of $\prod_{t\in X} A_t$ under pointwise algebraic operations.
(ii) For each $x \in F$, the function $t \mapsto \| x(t) \|$ is a continuous function defined on X.
(iii) For each $t \in X$, $\{x(t) \mid x \in F\}$ is dense in A_t. Moreover, if each A_t is a C^*-algebra, we also require that F is closed in pointwise multiplication and involution.

A vector field x is said to be continuous at t_0 with respect to F if for each $\varepsilon > 0$ there exists a neighborhood U of t_0 and $y \in F$ such that $\| x(t) - y(t) \| < \varepsilon$ for all $t \in U$. Such vector field x is said to be continuous with respect to F if x is continuous at every point $t \in X$ with respect to F.

Definition 3.1.2. Let X be a locally compact Hausdorff space and for each $t \in X$, we associate a C^*-algebra A_t.

A full algebra of operator fields A is a collection of vector fields such that:

(i) A is a * algebra under pointwise algebraic operation.

(ii) For each $x \in A$, the function $t \mapsto \| x(t) \|$ is continuous and vanishes at infinity. $(t \in X)$

(iii) For each $t \in X$, $\{x(t) \mid x \in A\}$ is dense in A_t.

(iv) A is complete with respect to the norm $\| x \| = \sup_{t \in X} \| x(t) \|$.

Remark:

1. A is a C^*-algebra with respect to the norm defined in (iv).
2. It is automatically that for each $t \in X$, $\{x(t) \mid x \in A\} = A_t$.
3. A itself is a continuity structure for $(X, \{A_t\})$.
4. The following conditions are equivalent:

(i) A is a maximal full algebra of operator fields.

(ii) $A = C_0(F)$ for some continuity structure F.

(iii) $A = C_0(A)$.

Where $C_0(F)$ denotes the full algebra consisting all the vector fields which are continuous with respect to F and vanishing at infinity.

3.2 Main Results

Lemma 3.2.1. Let A be a C^*-algebra such that:

(i) min-Primal(A) is τ_s closed in Primal$'(A)$.

(ii) For each primitive ideal P, there exists a unique minimal primal ideal I_P such that $I_P \subseteq P$.

Define a map Φ : Prim$(A) \longrightarrow$ min-Primal(A) by $\Phi(P) = I_P$. Then

(a) Φ is open and continuous (where Prim(A) is equipped with hull-kernel topology).

(b) The closed sets on min-Primal(A) are given by hull-kernel process.

Proof. (a)Prove by contradiction, suppose there exists a net $\{P_\lambda\}$ of primitive ideas which converges to a primitive ideal P but $\Phi(P_\lambda) \not\to \Phi(P)$. We can find a neighborhood N of $\Phi(P)$ and a subnet $\{P_{\lambda'}\}$ of $\{P_\lambda\}$ such that $\Phi(P_{\lambda'}) \notin N$. By τ_s-compactness of Primal(A), there exists a subnet $\{P_\alpha\}$ of $\{P_{\lambda'}\}$ such that $\Phi(P_\alpha)$ τ_s-converges to some primal ideal J. For each $a \in A$, $\| a + J \| = \lim_\alpha \| a + \Phi(P_\alpha) \| \geq \liminf_\alpha \| a + P_\alpha \| \geq \| a + P \|$. This shows $J \subseteq P$. By assumption (i), and the fact $\Phi(P_\alpha) \to J$ (w.r.t. τ topology), so J is minimal-primal. On the other hand, we have $J = \Phi(P)$, contradicting to $\Phi(P_{\lambda'}) \notin N$. This shows the continuity of the map Φ.

Let U be an open set in Prim(A) and write $U = \{P \in \text{Prim}(A) \mid J \not\subseteq P\}$ for some ideal J. Let $V = \{I \in \text{min-Primal}(A) \mid J \not\subseteq I\}$ which is open in min-Primal(A). We claim: $V = \Phi(U)$. Obviously we have $\Phi(U) \subseteq V$, so we need to prove the other side. Let $I \in V$, then there exists $P \in$ Primal(A) such that $I \subseteq P$ and $J \not\subseteq P$. $P \in U$ and clearly $I = \Phi(P)$.

(b) Let S a subset of min-Primal(A). Since hull ker(S) is τ_w closed and hence τ_s closed. (Here we denote ker(S)$=\cap\{I \mid I \in S\}$ and hull(I) $= \{J \in \text{min-Primal}(A) \mid I \subseteq J\}$.) So the τ_s closure of S (denoted by $\overline{S}$) is contained in hull ker(S). Conversely, we let $T := \Phi^{-1}(\overline{S})$ which is closed in Prim(A). Write $T = \{P \in$

$\text{Prim}(A) \mid K \subseteq P\}$ for some ideal K. Note that for each $I \in S$, we have $\{P \in \text{Prim}(A) \mid I \subseteq P\} = \Phi^{-1}(\{I\}) \subseteq \Phi^{-1}(\overline{S}) = \{P \in \text{Prim}(A) \mid K \subseteq P\}$ and hence $K \subseteq I$. Taking intersection over all such I, we have $K \subseteq \ker(S)$. So, if $J \in \text{hull}\ker(S)$, then $\Phi^{-1}(\{J\}) \subseteq T$ and hence J$\in \overline{S}$. □

Lemma 3.2.2. Let A be a C^*-algebra such that:
(i) min-Primal(A) is τ_s closed in Primal$'(A)$, and
(ii) For each primitive ideal P, there exists a unique minimal primal ideal I_P such that $I_P \subseteq P$.
Let f : min-Primal$(A) \mapsto \mathbb{C}$ be a bounded continuous function. Then for each $a \in A$, there exists uniquely $b \in A$ such that $b + I = f(I)(a + I)$.

Proof. Define g : Primal$(A) \mapsto \mathbb{C}$ by $g = f \circ \Phi$, where Φ is the map defined in Lemma 3.2.1. Clearly g is a bounded and continuous function defined on Prim(A). By Danus-Hofmann Theorem, there exists $b \in A$ such that $b + P = g(P)(a + P)$. Let $I \in$ min-Primal(A). Let P be an arbitrary primitive ideal containing I. Then $b - g(P)a \in P$, i.e. , $b - f(I)a \in P$. Taking intersection over all such P, we have $b - f(I)a \in I$.
If there exists two such b, say, b_1 and b_2 which satisfy the condition, then $b_1 - b_2 = (b_1 - f(I)a) - (b_2 - f(I)a) \in I$, for each minimal primal ideal I. But $\cap\{I \mid I \in \text{min-Primal}(A)\} = \{0\}$. This force to $b_1 = b_2$. □

With these two lemmas, we can start our discussion.
Let A be a C^*-algebra such that min-Primal(A) is τ_s closed in *Primal*$'(A)$. Then (min-Primal$(A), \tau_s)$ is locally compact, Hausdorff and will be served as a base space. For each $I \in$ min-Primal(A), we associate a fibre algebra $A_I = A/I$. For each $a \in A$, we define an operator field $\widehat{a}$ by $\widehat{a}(I) = a + I \in A_I$. Define $\widehat{A} = \{\widehat{a} \mid a \in A\}$.

Proposition 3.2.3. The above $\widehat{A}$ is a full algebra of operator fields for the system (min-Primal$(A), \{A_I\}$) in sense of J.M.G.Fell [3].

Proof. (i) Clearly $\widehat{A}$ is closed under pointwise algebraic operations.
(ii) For each $a \in A$, the function $I \mapsto \| \widehat{a}(I) \|$ is continuous and vanishes at infinity. Continuity is obvious and we only need to show the function vanishes at infinity. Let $\{I_\lambda\}$ be a net of minimal primal ideals which converges (τ_s) to a minimal primal ideal I. Let $\varepsilon > 0$ be given. Define $K_\varepsilon = \{I \in \text{min-Primal}(A) \mid \| \hat{a}(I) \| \geq \varepsilon\}$ which is closed and hence τ_s closed in Primal$'(A)$ (since min-Primal(A) is closed in Primal$'(A)$). Write $K_\varepsilon = K'_\varepsilon \cap \text{min-Primal}(A)$, where $K'_\varepsilon = \{I \in \text{Primal}(A) \mid \| a + I \| \geq \varepsilon\}$. As K'_ε is τ_s compact and contained in Primal$'(A)$, K_ε, being an intersection of a τ_s closed and a τ_s compact set, hence it is compact.
(iii) For each $I \in \text{min-Primal}(A)$, clearly $\{a + I \mid a \in A\}$ is dense in A_I, (and in fact equal to it).
(iv) The norm on $\hat{A}$ defined by $\| \hat{a} \| = \sup_I \| \hat{a}(I) \|$ is completed. Firstly, note that for each $a \in A$, $\| a \| = \| \hat{a} \|$. The inequality $\| a \| \geq \| \hat{a} \|$ is trival. Conversely, we choose a primitive ideal P such that $\| a \| = \| a + P \|$, then choose a minimal primal ideal I contained in P. We have $\| \hat{a} \| \geq \| \hat{a}(I) \| \geq \| a + P \| = \| a \|$. Now, suppose $\{a_n\}$ is a sequence in A such that $\sum_{n=1}^{\infty} \| \widehat{a_n} \| < +\infty$ and hence $\sum_{n=1}^{\infty} \| a_n \| < +\infty$. By completeness of A, we may define $y = \sum_{n=1}^{\infty} a_n$. Clearly $\sum_{n=1}^{\infty} \widehat{a_n}$ conveges to the operator field $\hat{y}$. □

Remark

In fact A is * isomorphic to $\hat{A}$.

Now, we come to the most important theorem in this section.

Theorem 3.2.4. Let A be a C^*-algebra such that min-Primal(A) is τ_s closed in

Primal$'(A)$. Let $\hat{A}$ be the full algebra of operator fields defined as above. then the following conditions are equivalent:

(i)$\hat{A}$ is a maximal full algebra of operator fields.

(ii)Each primitive ideal of A contains a unique minimal primal ideal.

Proof. (i)$\Rightarrow$(ii). Prove by contradiction, suppose there exists a primitive ideal P containing two distinct minimal primal ideals J_1 J_2. By Urysohn Lemma, we may choose a continuous function with compact support h : min-Primal$(A) \to [0, 1]$ such that $h(J_1) = 1$ and $h(J_2) = 0$. Choose $a \in A \backslash P$ and define an operator field $x = h \cdot \hat{a}$. Clearly x is continuous and vanishes at infinity, so $x \in \hat{A}$ by maximality of $\hat{A}$. Write $x = \hat{b}$ for some $b \in A$. Then $b + J_1 = x(J_1) = a + J_1$, so $a - b \in J_1 \subseteq P$. On the other hand, $b + J_2 = x(J_2) = J_2 \subseteq P$, therefore $a = (a - b) + b \in P$. Contradiction.

(ii)$\Rightarrow$(i). Let x be an operator field which is continuous and vanishes at infinity on min-Primal(A) with respect to the continuity structure $\hat{A}$. Let $\varepsilon > 0$ be given. Let $K = \{I \in \text{min-Primal}(A) \mid \| x(I) \| \geq \varepsilon/4\}$ which is compact. By continuity of x, for each $I \in K$, there exists an open neighborhood U_I of I and $y_I \in \hat{A}$ such that $\| (x - y_I)(t) \| < \varepsilon/4$ for all $t \in U_I$. By compactness of K, we may choose a finite covering of K, say, $U_1, U_2, \ldots, U_n$, with the corresponding $y_1, y_2, \ldots, y_n \in \hat{A}$. By partition of unity, there exists continuous functions $f_1, f_2, \ldots, f_n$ defined on min-Primal(A) such that $0 \leq f_k \leq 1$, the support of f_k being compact, contained in U_k , $\sum_{k=1}^{n} f_k \leq 1$ and $\sum_{k=1}^{n} f_k = 1$ on K. Define the operator field $y = \sum_{k=1}^{n} f_k \cdot y_k$. By Lemma 3.4, $y \in \hat{A}$. It is a routine to show $\| x(I) - y(I) \| < \varepsilon$ for all minimal primal ideal I. Hence $x \in \hat{A}$. $\square$

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